

# A NONFINITELY BASED SEMIGROUP OF TRIANGULAR MATRICES

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**ABSTRACT.** A new sufficient condition under which a semigroup admits no finite identity basis has been recently suggested in a joint paper by Karl Auinger, Yuzhu Chen, Xun Hu, Yanfeng Luo, and the author. Here we apply this condition to show the absence of a finite identity basis for the semigroup  $UT_3(\mathbb{R})$  of all upper triangular real  $3 \times 3$ -matrices with 0s and/or 1s on the main diagonal. The result holds also for the case when  $UT_3(\mathbb{R})$  is considered as an involution semigroup under the reflection with respect to the secondary diagonal.

## INTRODUCTION

A *semigroup identity* is just a pair of *words*, i.e., elements of the free semigroup  $A^+$  over an alphabet  $A$ . In this paper identities are written as “bumped” equalities such as  $u \simeq v$ . The identity  $u \simeq v$  is *trivial* if  $u = v$  and *nontrivial* otherwise. A semigroup  $S$  *satisfies*  $u \simeq v$  where  $u, v \in A^+$  if for every homomorphism  $\varphi : A^+ \rightarrow S$ , the equality  $u\varphi = v\varphi$  is valid in  $S$ ; alternatively, we say that  $u \simeq v$  *holds* in  $S$ . Clearly, trivial identities hold in every semigroup, and there exist semigroups (for instance, free semigroups over non-singleton alphabets) that satisfy only trivial identities.

Given any system  $\Sigma$  of semigroup identities, we say that an identity  $u \simeq v$  *follows* from  $\Sigma$  if every semigroup satisfying all identities of  $\Sigma$  satisfies the identity  $u \simeq v$  as well; alternatively, we say that  $\Sigma$  *implies*  $u \simeq v$ . A semigroup  $S$  is said to be *finitely based* if there exists a finite identity system  $\Sigma$  such that every identity holding in  $S$  follows from  $\Sigma$ ; otherwise  $S$  is called *nonfinitely based*.

The finite basis problem, that is, the problem of classifying semigroups according to the finite basability of their identities, has been intensively explored since the mid-1960s when the very first examples of nonfinitely based semigroups were discovered by Austin [3], Biryukov [5], and Perkins [15, 16]. One of the examples by Perkins was especially impressive as it involved a very transparent and natural object. Namely, Perkins proved that the finite basis property fails for the 6-element semigroup formed by the following integer  $2 \times 2$ -matrices under the usual matrix multiplication:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, even a finite semigroup can be nonfinitely based; moreover, it turns out that semigroups are the only “classical” algebras for which finite nonfinitely based objects can exist: finite groups [14], finite associative and Lie rings [10, 11, 4], finite lattices [13] are all finitely based. Therefore studying finite semigroups from the viewpoint of the finite basis problem has become a hot area in which many neat results have been achieved and several powerful methods have been developed, see the survey [18] for an overview.

It may appear surprising but the finite basis problem for **infinite** semigroups is less studied. The reason for this is that infinite semigroups usually arise in mathematics as semigroups of transformations of an infinite set, or semigroups of relations on an infinite domain, or semigroups matrices over an infinite ring, and as a rule all these semigroups are “too big” to satisfy any nontrivial identity. For instance (see, e.g., [6]), the two integer matrices

$$\begin{pmatrix} 2 & 0 \\ & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ & 1 \end{pmatrix}$$

are known to generate a free subsemigroup in the semigroup  $T_2(\mathbb{Z})$  of all upper triangular integer  $2 \times 2$ -matrices. (Here and below we omit zero entries under the main diagonal when dealing with upper triangular matrices.) Therefore even such a “small” matrix semigroup as  $T_2(\mathbb{Z})$  satisfies only trivial identities, to say nothing about matrix semigroups of larger dimension.

If all identities holding in a semigroup  $S$  are trivial,  $S$  is finitely based in a void way, so to speak. If, however, an infinite semigroup satisfies a nontrivial identity, its finite basis problem may constitute a challenge since “finite” methods are non-applicable in general. Therefore, up to recently, results classifying finitely based and nonfinitely based members within natural families of concrete infinite semigroups that contain semigroups with a nontrivial identity have been rather sparse.

Auinger, Chen, Hu, Luo, and the author [1] have found a new sufficient condition under which a semigroup is nonfinitely based and applied this condition to certain important classes of infinite semigroups. In the present paper we demonstrate yet another application; its interesting feature is that it requires the full strength of the main result of [1]. Namely, we prove that the semigroup  $UT_3(\mathbb{R})$  of all upper triangular real  $3 \times 3$ -matrices whose main diagonal entries are 0s and/or 1s is nonfinitely based. The result holds also for the case when  $UT_3(\mathbb{R})$  is considered as an involution semigroup under the reflection with respect to the secondary diagonal.

The paper is structured as follows. In Section 1 we recall the main result from [1], and in Section 2 we apply it to the semigroup  $UT_3(\mathbb{R})$ . Section 3 collects some concluding remarks and a related open question.

An effort has been made to keep this paper self-contained, to a reasonable extent. We use only the most basic concepts of semigroup theory and universal algebra that all can be found in the early chapters of the textbooks [7, 8], a suitable version of the main theorem from [1], and a few minor results from [2, 12, 17].

## 1. A SUFFICIENT CONDITION FOR THE NON-EXISTENCE OF A FINITE BASIS

The sufficient condition for the non-existence of a finite basis established in [1] applies to both plain semigroups, i.e., semigroups treated as algebras of type (2), and semigroups with involution as algebras of type (2,1). Let us recall all the concepts needed to formulate this condition.

We start with the definition of an involution semigroup. An algebra  $\langle S, \cdot, * \rangle$  of type (2,1) is called an *involution semigroup* if  $\langle S, \cdot \rangle$  is a semigroup (referred to as the *semigroup reduct* of  $\langle S, \cdot, * \rangle$ ) and the unary operation  $x \mapsto x^*$  is an involutory anti-automorphism of  $\langle S, \cdot \rangle$ , that is,

$$(xy)^* = y^*x^* \text{ and } (x^*)^* = x$$

for all  $x, y \in S$ .

The *free involution semigroup*  $\mathcal{FI}(A)$  on a given alphabet  $A$  can be constructed as follows. Let  $\overline{A} := \{a^* \mid a \in A\}$  be a disjoint copy of  $A$ . Define  $(a^*)^* := a$  for all  $a^* \in \overline{A}$ . Then  $\mathcal{FI}(A)$  is the free semigroup  $(A \cup \overline{A})^+$  endowed with the involution defined by

$$(a_1 \cdots a_m)^* := a_m^* \cdots a_1^*$$

for all  $a_1, \dots, a_m \in A \cup \overline{A}$ . We refer to elements of  $\mathcal{FI}(A)$  as *involutory words over  $A$* . An *involutory identity*  $u \simeq v$  is just a pair of involutory words; the identity *holds* in an involution semigroup  $S$  if for every involution semigroup homomorphism  $\varphi : \mathcal{FI}(A)^+ \rightarrow S$ , the equality  $u\varphi = v\varphi$  is valid in  $S$ . Now the concepts of a finitely based/nonfinitely based involution semigroup are defined exactly as in the plain semigroup case. In what follows, we use square brackets to indicate adjustments to be made in the involution case.

A class  $\mathbf{V}$  of [involution] semigroups is called a *variety* if there exists a system  $\Sigma$  of [involution] semigroup identities such that  $\mathbf{V}$  consists precisely of all [involution] semigroups that satisfy every identity in  $\Sigma$ . In this case we say that  $\mathbf{V}$  is *defined* by  $\Sigma$ . If the system  $\Sigma$  can be chosen to be finite, the corresponding variety is said to be *finitely based*; otherwise it is *nonfinitely based*. Given a class  $\mathbf{K}$  of [involution] semigroups, the variety defined by the identities that hold in each [involution] semigroup from  $\mathbf{K}$  is said to be *generated by  $\mathbf{K}$*  and is denoted by  $\text{var } \mathbf{K}$ ; if  $\mathbf{K} = \{S\}$ , we write  $\text{var } S$  rather than  $\text{var } \{S\}$ . It should be clear that  $S$  and  $\text{var } S$  are simultaneously finitely based or nonfinitely based.

A semigroup is said to be *periodic* if each of its one-generated subsemigroups is finite and *locally finite* if each of its finitely generated subsemigroups is finite. A variety of semigroups is *locally finite* if all its members are locally finite.

Let  $\mathbf{A}$  and  $\mathbf{B}$  be two classes of semigroups. The *Mal'cev product*  $\mathbf{A} \circledast \mathbf{B}$  of  $\mathbf{A}$  and  $\mathbf{B}$  is the class of all semigroups  $S$  for which there exists a congruence  $\theta$  such that the quotient semigroup  $S/\theta$  lies in  $\mathbf{B}$  while all  $\theta$ -classes that are subsemigroups in  $S$  belong to  $\mathbf{A}$ . Notice that for a congruence  $\theta$  on a semigroup  $S$ , a  $\theta$ -class forms a subsemigroup of  $S$  if and only if the class is an idempotent of the quotient semigroup  $S/\theta$ .

Let  $x_1, x_2, \dots, x_n, \dots$  be a sequence of letters. The sequence  $\{Z_n\}_{n=1,2,\dots}$  of *Zimin words* is defined inductively by  $Z_1 := x_1$ ,  $Z_{n+1} := Z_n x_{n+1} Z_n$ . We say that a word  $v$  is an *[involutory] isoterms* for a class  $\mathbf{C}$  of semigroups [with involution] if the only [involutory] word  $v'$  such that all members of  $\mathbf{C}$  satisfy the [involution] semigroup identity  $v \simeq v'$  is the word  $v$  itself.

Now we are in a position to state the main result of [1]. Here  $\mathbf{Com}$  denotes the variety of all commutative semigroups.

**Theorem 1** ([1, Theorem 6]). *A variety  $\mathbf{V}$  of [involution] semigroups is nonfinitely based provided that*

- (i) *[the class of all semigroup reducts of]  $\mathbf{V}$  is contained in the variety  $\text{var}(\mathbf{Com} \circledast \mathbf{W})$  for some locally finite semigroup variety  $\mathbf{W}$  and*
- (ii) *each Zimin word is an [involutory] isoterms relative to  $\mathbf{V}$ .*

Formulated as above, Theorem 1 suffices for all applications presented in [1] but is insufficient for the purposes of the present paper. However, it is observed in [1, Remark 1] that the theorem remains valid if one replaces the condition (i) by the following weaker condition:

- (i') *[the class of all semigroup reducts of]  $\mathbf{V}$  is contained in the variety  $\text{var}(\mathbf{U} \circledast \mathbf{W})$  where  $\mathbf{U}$  is a semigroup variety all of whose periodic members are locally finite and  $\mathbf{W}$  is a locally finite semigroup variety.*

Here we will utilize this stronger form of Theorem 1.

## 2. THE IDENTITIES OF $\text{UT}_3(\mathbb{R})$

Recall that we denote by  $\text{UT}_3(\mathbb{R})$  the semigroup of all upper triangular real  $3 \times 3$ -matrices whose main diagonal entries are 0s and/or 1s. For each matrix  $\alpha \in \text{UT}_3(\mathbb{R})$ , let  $\alpha^D$  stand for the matrix obtained by reflecting  $\alpha$  with respect to the secondary diagonal (from the top right to the bottom left corner); in other words,  $(\alpha_{ij})^D := (\alpha_{4-j, 4-i})$ . Then it is easy to verify that the unary operation  $\alpha \mapsto \alpha^D$  (called the *skew transposition*) is an involutory anti-automorphism of  $\text{UT}_3(\mathbb{R})$ . Thus, we can consider  $\text{UT}_3(\mathbb{R})$  also as an involution semigroup. Our main result is the following

**Theorem 2.** *The semigroup  $\text{UT}_3(\mathbb{R})$  is nonfinitely based as both a plain semigroup and an involution semigroup under the skew transposition.*

*Proof.* We will verify that the [involution] semigroup variety  $\text{var } \text{UT}_3(\mathbb{R})$  satisfies the conditions (i') and (ii) discussed at the end of Section 1; the desired result will then follow from Theorem 1 in its stronger form.

Let  $D_3$  denote the 8-element subsemigroup of  $\text{UT}_3(\mathbb{R})$  consisting of all diagonal matrices. To every matrix  $\alpha \in \text{UT}_3(\mathbb{R})$  we assign the diagonal matrix  $\text{Diag}(\alpha) \in D_3$  by changing each non-diagonal entry of  $\alpha$  to 0. The following observation is obvious.

**Lemma 3.** *The map  $\alpha \mapsto \text{Diag}(\alpha)$  is a homomorphism of  $\text{UT}_3(\mathbb{R})$  onto  $D_3$ .*

We denote by  $\theta$  the kernel of the homomorphism of Lemma 3, i.e.,

$$(\alpha, \beta) \in \theta \text{ if and only if } \text{Diag}(\alpha) = \text{Diag}(\beta).$$

Then  $\theta$  is a congruence on  $\text{UT}_3(\mathbb{R})$ . Since each element of the semigroup  $D_3$  is an idempotent, each  $\theta$ -class is a subsemigroup of  $\text{UT}_3(\mathbb{R})$ . The next fact is the core of our proof.

**Proposition 4.** *Each  $\theta$ -class of  $\text{UT}_3(\mathbb{R})$  satisfies the identity*

$$Z_4 \simeq (x_1 x_2)^2 x_1 x_3 x_1 x_4 x_1 x_3 x_1 (x_2 x_1)^2. \quad (1)$$

*Proof.* We have to consider 8 cases. First we observe that the identity (1) is left-right symmetric, and therefore, (1) holds in some subsemigroup  $S$  of  $\text{UT}_3(\mathbb{R})$  if and only if it holds in the subsemigroup  $S^D = \{s^D \mid s \in S\}$  since  $S^D$  is anti-isomorphic to  $S$ . This helps us to shorten the below analysis.

**Case 1:**  $S_{000} = \left\{ \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ & 0 & \alpha_{23} \\ & & 0 \end{pmatrix} \mid \alpha_{12}, \alpha_{13}, \alpha_{23} \in \mathbb{R} \right\}$ . This subsemigroup is easily seen to satisfy the identity  $x_1 x_2 x_3 \simeq y_1 y_2 y_3$  which clearly implies (1).

**Case 2:**  $S_{100} = \left\{ \begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} \\ & 0 & \alpha_{23} \\ & & 0 \end{pmatrix} \mid \alpha_{12}, \alpha_{13}, \alpha_{23} \in \mathbb{R} \right\}$ . Multiplying 3 arbitrary matrices  $\alpha, \beta, \gamma \in S_{100}$ , we get

$$\begin{aligned} \begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} \\ & 0 & \alpha_{23} \\ & & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta_{12} & \beta_{13} \\ & 0 & \beta_{23} \\ & & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \gamma_{12} & \gamma_{13} \\ & 0 & \gamma_{23} \\ & & 0 \end{pmatrix} = \\ \begin{pmatrix} 1 & \gamma_{12} & \gamma_{13} + \beta_{12}\gamma_{23} \\ & 0 & 0 \\ & & 0 \end{pmatrix} = \begin{pmatrix} 1 & \beta_{12} & \beta_{13} \\ & 0 & \beta_{23} \\ & & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \gamma_{12} & \gamma_{13} \\ & 0 & \gamma_{23} \\ & & 0 \end{pmatrix}. \end{aligned}$$

Thus,  $\alpha\beta\gamma = \beta\gamma$  and we have proved that  $S_{100}$  satisfies the identity  $xyz \simeq yz$ . Clearly, this identity implies (1).

**Case 3:**  $S_{010} = \left\{ \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ & 1 & \alpha_{23} \\ & & 0 \end{pmatrix} \mid \alpha_{12}, \alpha_{13}, \alpha_{23} \in \mathbb{R} \right\}$ . It is easy to see that this subsemigroup satisfies the identity  $xyx \simeq x$  which clearly implies (1).

**Case 4:**  $S_{001} = \left\{ \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ & 0 & \alpha_{23} \\ & & 1 \end{pmatrix} \mid \alpha_{12}, \alpha_{13}, \alpha_{23} \in \mathbb{R} \right\}$ . This case reduces to Case 2 since  $S_{001} = S_{100}^D$ .

**Case 5:**  $S_{110} = \left\{ \begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} \\ & 1 & \alpha_{23} \\ & & 0 \end{pmatrix} \mid \alpha_{12}, \alpha_{13}, \alpha_{23} \in \mathbb{R} \right\}$ . Multiplying 3 arbitrary matrices  $\alpha, \beta, \gamma \in S_{110}$ , we get

$$\begin{aligned} \begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} \\ & 1 & \alpha_{23} \\ & & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta_{12} & \beta_{13} \\ & 1 & \beta_{23} \\ & & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \gamma_{12} & \gamma_{13} \\ & 1 & \gamma_{23} \\ & & 0 \end{pmatrix} = \\ \begin{pmatrix} 1 & \alpha_{12} + \beta_{12} + \gamma_{12} & \gamma_{13} + (\alpha_{12} + \beta_{12})\gamma_{23} \\ & 1 & \gamma_{23} \\ & & 0 \end{pmatrix} \end{aligned}$$

whence the product  $\alpha\beta\gamma$  depends only on  $\gamma$  and on the sum  $\alpha_{12} + \beta_{12}$ . Thus,  $\alpha\beta\gamma = \beta\alpha\gamma$  and we have proved that  $S_{110}$  satisfies the identity  $xyz \simeq yxz$ . This identity implies (1).

**Case 6:**  $S_{101} = \left\{ \begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{23} & 1 \end{pmatrix} \mid \alpha_{12}, \alpha_{13}, \alpha_{23} \in \mathbb{R} \right\}$ . Take an arbitrary homomorphism  $\varphi : \{x_1, x_2, x_3, x_4\}^+ \rightarrow S_{101}$  and let  $\alpha = x_1\varphi$ ,  $\beta = x_2\varphi$ ,  $\gamma = x_3\varphi$ , and  $\delta = x_4\varphi$ . Then one can calculate that both  $Z_4\varphi$  and  $(x_1x_2x_1x_2x_1x_3x_1x_4x_1x_3x_1x_2x_1x_2x_1)\varphi$  are equal to the matrix  $\begin{pmatrix} 1 & \alpha_{12} & \varepsilon \\ 0 & \alpha_{23} & 1 \end{pmatrix}$  where  $\varepsilon$  stands for the following expression:

$$8\alpha_{13} + 4\beta_{13} + 2\gamma_{13} + \delta_{13} + \alpha_{12}(4\beta_{23} + 2\gamma_{23} + \delta_{23}) + (4\beta_{12} + 2\gamma_{12} + \delta_{12})\alpha_{23}.$$

Thus, the identity (1) holds on  $S_{101}$ .

For readers familiar with the Rees matrix construction (cf. [8, Chapter 3]), we outline a more conceptual proof for the fact that  $S_{101}$  satisfies (1). Let  $G = \langle \mathbb{R}, + \rangle$  stand for the additive group of real numbers and let  $P$  be the  $\mathbb{R} \times \mathbb{R}$ -matrix over  $G$  whose element in the  $r$ th row and the  $s$ th column is equal to  $rs$ . One readily verifies that the map  $\begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{23} & 1 \end{pmatrix} \mapsto (\alpha_{23}, \alpha_{13}, \alpha_{12})$  constitutes an isomorphism of the semigroup  $S_{101}$  onto the Rees matrix semigroup  $M(\mathbb{R}, G, \mathbb{R}; P)$ . It is known (see, e.g., [9]) and easy to verify that every Rees matrix semigroup over an Abelian group satisfies each identity  $u \simeq v$  for which the following three conditions hold: the first letter of  $u$  is the same as the first letter of  $v$ ; the last letter of  $u$  is the same as the last letter of  $v$ ; for each ordered pair of letters the number of occurrences of this pair is the same in  $u$  and  $v$ . Inspecting the identity (1), one immediately sees that it satisfies the three conditions whence it holds in the semigroup  $M(\mathbb{R}, G, \mathbb{R}; P)$  and also in the semigroup  $S_{101}$  isomorphic to  $M(\mathbb{R}, G, \mathbb{R}; P)$ .

**Case 7:**  $S_{011} = \left\{ \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ 1 & \alpha_{23} & 1 \end{pmatrix} \mid \alpha_{12}, \alpha_{13}, \alpha_{23} \in \mathbb{R} \right\}$ . This case reduces to Case 5 since  $S_{011} = S_{110}^D$ .

**Case 8:**  $S_{111} = \left\{ \begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} \\ 1 & \alpha_{23} & 1 \end{pmatrix} \mid \alpha_{12}, \alpha_{13}, \alpha_{23} \in \mathbb{R} \right\}$ . The semigroup  $S_{111}$  is in fact the group of all real upper unitriangular  $3 \times 3$ -matrices. The latter group is known to be nilpotent of class 2, and as observed by Mal'cev [12], every nilpotent group of class 2 satisfies the semigroup identity

$$xzytyzx \simeq yzxtxzy. \quad (2)$$

Now we verify that (1) follows from (2). For this, we substitute in (2) the letter  $x_1$  for  $x$ , the letter  $x_3$  for  $z$ , the word  $x_1x_2x_1$  for  $y$ , and the letter  $x_4$  for  $t$ . We then obtain the identity

$$\underbrace{x_1}_x \underbrace{x_3}_z \underbrace{x_1x_2x_1}_y \underbrace{x_4}_t \underbrace{x_1x_2x_1}_y \underbrace{x_3}_z \underbrace{x_1}_x \simeq \underbrace{x_1x_2x_1}_y \underbrace{x_3}_z \underbrace{x_1}_x \underbrace{x_4}_t \underbrace{x_1}_x \underbrace{x_3}_z \underbrace{x_1x_2x_1}_y.$$

Multiplying this identity through by  $x_1x_2$  on the left and by  $x_2x_1$  on the right, we get (1).  $\square$

Recall that a semigroup identity  $u \simeq v$  is said to be *balanced* if for every letter the number of occurrences of this letter is the same in  $u$  and  $v$ . Clearly, the identity (1) is balanced.

**Lemma 5** ([17, Lemma 3.3]). *If a semigroup variety  $\mathbf{V}$  satisfies a nontrivial balanced identity of the form  $Z_n \simeq v$ , then all periodic members of  $\mathbf{V}$  are locally finite.*

Let  $\mathbf{U}$  stand for the semigroup variety defined by the identity (1). Then Lemma 5 ensures that all periodic members of  $\mathbf{U}$  are locally finite while Lemma 3 and Proposition 4 imply that the semigroup  $\text{UT}_3(\mathbb{R})$  lies in the Mal'cev product  $\mathbf{U} \circledast \text{var } \mathbf{D}_3$ . The variety  $\text{var } \mathbf{D}_3$  is locally finite as a variety generated by a finite semigroup [7, Theorem 10.16]. We see that the variety  $\text{var } \text{UT}_3(\mathbb{R})$  satisfies the condition (i').

It remains to verify that  $\text{var } \text{UT}_3(\mathbb{R})$  satisfies the condition (ii) as well. Clearly, the involutory version of the condition (ii) is stronger than its plain version so that it suffices to show that each Zimin word is an involutory isoterm relative to  $\text{UT}_3(\mathbb{R})$  considered as an involution semigroup.

Let  $\text{TA}_2^1$  stand for the involution semigroup formed by the  $(0, 1)$ -matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

under the usual matrix multiplication and the unary operation that swaps each of the matrices  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  with the other one and fixes the rest four matrices. By [2, Corollaries 2.7 and 2.8] each Zimin word is an involutory isoterm relative to  $\text{TA}_2^1$ . Now consider the involution subsemigroup  $M$  in  $\text{UT}_3(\mathbb{R})$  generated by the matrices

$$e = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \quad x = \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}, \quad \text{and} \quad y = \begin{pmatrix} 1 & 1 & 0 \\ & 0 & 1 \\ & & 1 \end{pmatrix}.$$

Clearly, for each matrix  $(\mu_{ij}) \in M$ , one has  $\mu_{ij} \geq 0$  and  $\mu_{11} = \mu_{33} = 1$ , whence the set  $N$  of all matrices  $(\mu_{ij}) \in M$  such that  $\mu_{13} > 0$  forms an ideal in  $M$ . Clearly,  $N$  is closed under the skew transposition. A straightforward calculation shows that, besides  $e$ ,  $x$ , and  $y$ , the only matrices in  $M \setminus N$  are  $xy = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  and  $yx = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ . Consider the following bijection between  $M \setminus N$  and the set of non-zero matrices in  $\text{TA}_2^1$ :

$$e \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad xy \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad yx \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Extending this bijection to  $M$  by sending all elements from  $N$  to  $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$  yields an involution semigroup homomorphism from  $M$  onto  $\text{TA}_2^1$ . Thus,  $\text{TA}_2^1$  as a homomorphic image of an involution subsemigroup in  $\text{UT}_3(\mathbb{R})$  satisfies all involution semigroup identities that hold in  $\text{UT}_3(\mathbb{R})$ . Therefore each Zimin word is an involutory isoterm relative to  $\text{UT}_3(\mathbb{R})$ , as required.  $\square$

### 3. CONCLUDING REMARKS AND AN OPEN QUESTION

Here we discuss which conditions of Theorem 2 are essential and which can be relaxed.

It should be clear from the above proof of Theorem 2 that the fact that we have dealt with matrices over the field  $\mathbb{R}$  is not really essential: the proof works for every semigroup of the form  $\text{UT}_3(R)$  where  $R$  is an arbitrary associative and commutative ring with 1 such that

$$\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} \neq 0 \quad (3)$$

for every positive integer  $n$ . For instance, we can conclude that the semigroup  $\text{UT}_3(\mathbb{Z})$  of all upper triangular integer  $3 \times 3$ -matrices whose main diagonal entries are 0s and/or 1s is nonfinitely based in both plain and involution semigroup settings.

On the other hand, we cannot get rid of the restriction imposed on the main diagonal entries: as the example reproduced in the introduction implies, the semigroup  $\text{T}_3(\mathbb{Z})$  of all upper triangular integer  $3 \times 3$ -matrices is finitely based as a plain semigroup since it satisfies only trivial semigroup identities. In a similar way one can show that  $\text{T}_3(\mathbb{Z})$  is finitely based when considered as an involution semigroup with the skew transposition. Indeed, the subsemigroup generated in  $\text{T}_3(\mathbb{Z})$  by the matrix  $\zeta = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$  and its skew transpose  $\zeta^D$  is free, and therefore, considered as an involution semigroup, it is isomorphic to the free involution semigroup on one generator, say  $z$ . However  $\mathcal{FI}(\{z\})$  contains as an involution subsemigroup a free involutory semigroup on countably many generators, namely,  $\mathcal{FI}(Z)$  where

$$Z = \{zz^*z, z(z^*)^2z, \dots, z(z^*)^nz, \dots\}.$$

Hence  $\text{T}_3(\mathbb{Z})$  satisfies only trivial involution semigroup identities. Of course, the same conclusions persist if we substitute  $\mathbb{Z}$  by any associative and commutative ring with 1 satisfying (3) for every  $n$ .

We can however slightly weaken the restriction on the main diagonal entries by allowing them to take values in the set  $\{0, \pm 1\}$ . The proof of Theorem 2 remains valid for the resulting [involution] semigroup that we denote by  $\text{UT}_3^\pm(\mathbb{R})$ . Indeed, the homomorphism  $\alpha \mapsto \text{Diag}(\alpha)$  of Lemma 3 extends to a homomorphism of  $\text{UT}_3^\pm(\mathbb{R})$  onto its 27-element subsemigroup consisting of diagonal matrices. The subsemigroup classes of the kernel of this homomorphism are precisely the subsemigroups  $S_{000}, \dots, S_{111}$  from the proof of Proposition 4, and therefore, the variety  $\text{var } \text{UT}_3^\pm(\mathbb{R})$  satisfies the condition (i') of the stronger form of Theorem 1. Of course, the variety fulfils also the condition (ii) since (ii) is inherited by supervarieties. In the same fashion, the proof of Theorem 2 applies, say, to the semigroup of all upper triangular complex  $3 \times 3$ -matrices whose main diagonal entries come from the set  $\{0, 1, \xi, \dots, \xi^{n-1}\}$  where  $\xi$  is a primitive  $n$ th root of unity.



The question of whether or not a result similar to Theorem 2 holds true for analogs of the semigroup  $\text{UT}_3(\mathbb{R})$  in other dimensions is more involved. The variety  $\text{var UT}_2(\mathbb{R})$  fulfils the condition (i') since the condition is clearly inherited by subvarieties and the injective map  $\text{UT}_2(\mathbb{R}) \rightarrow \text{UT}_3(\mathbb{R})$  defined by  $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ & \alpha_{22} \end{pmatrix} \mapsto \begin{pmatrix} \alpha_{11} & 0 & \alpha_{12} \\ & 0 & 0 \\ & & \alpha_{22} \end{pmatrix}$  is an embedding of [involution] semigroups whence  $\text{var UT}_2(\mathbb{R}) \subseteq \text{var UT}_3(\mathbb{R})$ . However,  $\text{var UT}_2(\mathbb{R})$  does not satisfy the condition (ii) as the following result shows.

**Proposition 6.** *The semigroup  $\text{UT}_2(\mathbb{R})$  of all upper triangular real  $2 \times 2$ -matrices whose main diagonal entries are 0s and/or 1s satisfies the identity*

$$Z_4 \simeq x_1 x_2 x_1 x_3 x_1^2 x_2 x_4 x_2 x_1^2 x_3 x_1 x_2 x_1. \quad (4)$$

*Proof.* Fix an arbitrary homomorphism  $\varphi : \{x_1, x_2, x_3, x_4\}^+ \rightarrow \text{UT}_2(\mathbb{R})$ . For brevity, denote the right hand side of (4) by  $w$ ; we thus have to prove that  $Z_4\varphi = w\varphi$ . Let

$$x_1\varphi = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ & \alpha_{22} \end{pmatrix}, \quad x_2\varphi = \begin{pmatrix} \beta_{11} & \beta_{12} \\ & \beta_{22} \end{pmatrix}, \quad x_3\varphi = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ & \gamma_{22} \end{pmatrix}, \quad x_4\varphi = \begin{pmatrix} \delta_{11} & \delta_{12} \\ & \delta_{22} \end{pmatrix},$$

where  $\alpha_{11}, \alpha_{22}, \beta_{11}, \beta_{22}, \gamma_{11}, \gamma_{22}, \delta_{11}, \delta_{22} \in \{0, 1\}$  and  $\alpha_{12}, \beta_{12}, \gamma_{12}, \delta_{12} \in \mathbb{R}$ .

If  $\alpha_{22} = 0$ , the fact that  $\alpha_{11}, \beta_{11}, \gamma_{11}, \delta_{11} \in \{0, 1\}$  readily implies that  $Z_4\varphi = w\varphi = \begin{pmatrix} \varepsilon & \varepsilon\alpha_{12} \\ & 0 \end{pmatrix}$ , where  $\varepsilon = \alpha_{11}\beta_{11}\gamma_{11}\delta_{11}$ . Similarly, if  $\alpha_{11} = 0$ , then it is easy to calculate that  $Z_4\varphi = w\varphi = \begin{pmatrix} 0 & \alpha_{12}\eta \\ & \eta \end{pmatrix}$ , where  $\eta = \alpha_{22}\beta_{22}\gamma_{22}\delta_{22}$ . We thus may (and will) assume that  $\alpha_{11} = \alpha_{22} = 1$ .

Now, if  $\beta_{22} = 0$ , a straightforward calculation shows that  $Z_4\varphi = w\varphi = \begin{pmatrix} \varkappa & \varkappa(\alpha_{12} + \beta_{12}) \\ & 0 \end{pmatrix}$ , where  $\varkappa = \beta_{11}\gamma_{11}\delta_{11}$ . Similarly, if  $\beta_{11} = 0$ , we get  $Z_4\varphi = w\varphi = \begin{pmatrix} 0 & (\alpha_{12} + \beta_{12})\lambda \\ & \lambda \end{pmatrix}$ , where  $\lambda = \beta_{22}\gamma_{22}\delta_{22}$ . Thus, we may also assume that  $\beta_{11} = \beta_{22} = 1$ . Observe that the word  $w$  is obtained from the word  $Z_4$  by substituting  $x_1^2 x_2$  for the second occurrence of the factor  $x_1 x_2 x_1$  and  $x_2 x_1^2$  for the third occurrence of this factor. Therefore  $\alpha_{11} = \alpha_{22} = \beta_{11} = \beta_{22} = 1$  implies  $x_1 x_2 x_1 \varphi = x_1^2 x_2 \varphi = x_2 x_1^2 \varphi = \begin{pmatrix} 1 & 2\alpha_{12} + \beta_{12} \\ & 1 \end{pmatrix}$  whence  $Z_4\varphi = w\varphi$ .  $\square$

Now let  $\text{UT}_n(\mathbb{R})$  stand for the semigroup of all upper triangular real  $n \times n$ -matrices whose main diagonal entries are 0s and/or 1s and assume that  $n \geq 4$ . Here the behaviour of the [involution] semigroup variety generated by  $\text{UT}_n(\mathbb{R})$  with respect to the conditions of Theorem 1 is in a sense opposite. Namely, it is not hard to show (by using an argument similar to the one utilized in the proof of Theorem 2) that the variety  $\text{var UT}_n(\mathbb{R})$  with  $n \geq 4$  satisfies the condition (ii). On the other hand, the approach used in the proof of Theorem 2 fails to justify that this variety fulfils (i'). In order to explain this claim, suppose for simplicity that  $n = 4$ . Then the homomorphism  $\alpha \mapsto \text{Diag}(\alpha)$  maps  $\text{UT}_4(\mathbb{R})$  onto its 16-element subsemigroup consisting of diagonal matrices which all are idempotent. This induces a partition of  $\text{UT}_4(\mathbb{R})$  into 16 subsemigroups, and to mimic the proof of Theorem 2 one should show that all these subsemigroups belong to a variety whose periodic

members are locally finite. One of these 16 subsemigroups is nothing but the group of all real upper unitriangular  $4 \times 4$ -matrices. The latter group is known to be nilpotent of class 3, and one might hope to use the identity

$$xzytyzxsyzxtxzy \simeq yzxtxzysxzytyzx, \quad (5)$$

proved by Mal'cev [12] to hold in every nilpotent group of class 3, along the lines of the proof of Proposition 4 where we have invoked Mal'cev's identity holding in each nilpotent group of class 2. However, it is known [19, Theorem 2] that the variety defined by (5) contains infinite finitely generated periodic semigroups. Even though this fact does not yet mean that the condition (i') fails in  $\text{var UT}_4(\mathbb{R})$ , it demonstrates that the techniques presented in this paper are not powerful enough to verify whether or not the variety obeys this condition. It seems that this verification constitutes a very difficult task as it is closely connected with Sapir's longstanding conjecture that for each nilpotent group  $G$ , periodic members of the semigroup variety  $\text{var } G$  are locally finite, see [17, Section 5].

Back to our discussion, we see that Theorem 1 cannot be applied to the semigroup  $\text{UT}_2(\mathbb{R})$  and we are not in a position to apply it to the semigroups  $\text{UT}_n(\mathbb{R})$  with  $n \geq 4$ . Of course, this does not indicate that these semigroups are finitely based—recall that Theorem 1 is only a sufficient condition for being nonfinitely based. Presently, we do not know which of the semigroups  $\text{UT}_n(\mathbb{R})$  with  $n \neq 3$  possess the finite basis property, and we conclude the paper with explicitly stating this open question in the anticipation that, over time, looking for an answer might stimulate creating new approaches to the finite basis problem for infinite [involution] semigroups:

*Question.* For which  $n \neq 3$  is the [involution] semigroup  $\text{UT}_n(\mathbb{R})$  finitely based?

**Acknowledgements.** The author gratefully acknowledges support from the Presidential Programme “Leading Scientific Schools of the Russian Federation”, project no. 5161.2014.1, and from the Russian Foundation for Basic Research, project no. 14-01-00524. In particular, the latter project was used to support the author's participation in the International Conference on Semigroups, Algebras and Operator Theory (ICSAOT-2014) held at the Cochin University of Science and Technology as well as his participation in the International Conference on Algebra and Discrete Mathematics (ICADM-2014) held at Government College, Kattappana. Also the author expresses cordial thanks to the organizers of these two conferences, especially to Dr. P. G. Romeo, Convenor of ICSAOT-2014, and Sri. G. N. Prakash, Convenor of ICADM-2014, for their warm hospitality.

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